数理科学与大数据本科生2021-2022学年第二学期

“数学分析II”期中考试试题参考解答

一、(15分) 判断极限$\lim\limits_{x^2+y^2\to+\infty}\dfrac{x^2+y^2}{x^2y^2}$是否存在，如果存在并求其值.

解 极限$\lim\limits_{x^2+y^2\to+\infty}\dfrac{x^2+y^2}{x^2y^2}$不存在. 证明如下. 取$(x_m,y_m)=\left(m,\dfrac{1}{m}\right)$, $m=1,2,\cdots$, 则$x_m^2+y_m^2=m^2+\dfrac{1}{m^2}\to+\infty\ (m\to\infty)$, 而

$$\lim_{m\to\infty}\frac{x_m^2+y_m^2}{x_m^2y_m^2}=\lim_{m\to\infty}\frac{m^2+\frac{1}{m^2}}{m^2\cdot\frac{1}{m^2}}=\lim_{m\to\infty}\left(m^2+\frac{1}{m^2}\right)=+\infty,$$

故极限$\lim\limits_{x^2+y^2\to+\infty}\dfrac{x^2+y^2}{x^2y^2}$不存在. □

二、(15分) 设$f(x,y)=\ln(\mathrm{e}^x+\mathrm{e}^y)$, 求全微分$\mathrm{d}f(0,0)$和二阶全微分$\mathrm{d}^2f(0,0)$.

解 对$f(x,y)$求偏导，得

$$\frac{\partial f}{\partial x}=\frac{\mathrm{e}^x}{\mathrm{e}^x+\mathrm{e}^y},\quad \frac{\partial f}{\partial y}=\frac{\mathrm{e}^y}{\mathrm{e}^x+\mathrm{e}^y},$$

故

$$\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=\frac{1}{2},$$

于是

$$\mathrm{d}f(0,0)=\frac{\partial f}{\partial x}(0,0)\mathrm{d}x+\frac{\partial f}{\partial y}(0,0)\mathrm{d}y=\frac{1}{2}\mathrm{d}x+\frac{1}{2}\mathrm{d}y.$$

再对$f(x,y)$求二阶偏导，得

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\mathrm{e}^x\cdot(\mathrm{e}^x+\mathrm{e}^y)-\mathrm{e}^x\cdot\mathrm{e}^x}{(\mathrm{e}^x+\mathrm{e}^y)^2}=\frac{\mathrm{e}^x\mathrm{e}^y}{(\mathrm{e}^x+\mathrm{e}^y)^2},\\
\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{-\mathrm{e}^x\mathrm{e}^y}{(\mathrm{e}^x+\mathrm{e}^y)^2},\\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\mathrm{e}^y\cdot(\mathrm{e}^x+\mathrm{e}^y)-\mathrm{e}^y\cdot\mathrm{e}^y}{(\mathrm{e}^x+\mathrm{e}^y)^2}=\frac{\mathrm{e}^x\mathrm{e}^y}{(\mathrm{e}^x+\mathrm{e}^y)^2},
\end{aligned}$$

故

$$\frac{\partial^2 f}{\partial x^2}(0,0)=\frac{\partial^2 f}{\partial y^2}(0,0)=\frac{1}{4},\quad \frac{\partial^2 f}{\partial x\partial y}(0,0)=\frac{\partial^2 f}{\partial y\partial x}(0,0)=-\frac{1}{4},$$

于是

$$\begin{aligned}
\mathrm{d}^2f(0,0) &= \frac{\partial^2 f}{\partial x^2}(0,0)\mathrm{d}x^2+\frac{\partial^2 f}{\partial x\partial y}(0,0)\mathrm{d}x\mathrm{d}y+\frac{\partial^2 f}{\partial y\partial x}(0,0)\mathrm{d}x\mathrm{d}y+\frac{\partial^2 f}{\partial y^2}(0,0)\mathrm{d}y^2\\
&= \frac{1}{4}\mathrm{d}x^2-\frac{1}{2}\mathrm{d}x\mathrm{d}y+\frac{1}{4}\mathrm{d}y^2.
\end{aligned}$$

□

三、(15分) 求旋轮线$x=\sqrt{3}(t-\sin t)$, $y=\sqrt{3}(1-\cos t)$, $0\leqslant t\leqslant 2\pi$绕$x$轴旋转所得曲面的面积.

解 由旋转曲面的面积公式，得

$$\begin{aligned} S &= 2\pi\int_0^{2\pi} y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\mathrm{d}t \\ &= 2\pi\int_0^{2\pi}\sqrt{3}(1-\cos t)\sqrt{3(2-2\cos t)}\mathrm{d}t \\ &= 24\pi\int_0^{2\pi}\sin^3\frac{t}{2}\mathrm{d}t = 48\pi\int_0^{\pi}\sin^3 u\mathrm{d}u = 96\pi\int_0^{\frac{\pi}{2}}\sin^3 u\mathrm{d}u \\ &= 96\pi\cdot\frac{2}{3} = 64\pi. \end{aligned}$$

□

四、(15分) 计算定积分$\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x\arcsin x+\sin x}{\sqrt{1-x^2}}\mathrm{d}x$.

解 令$f(x)=\frac{x\arcsin x}{\sqrt{1-x^2}}$, $g(x)=\frac{\sin x}{\sqrt{1-x^2}}$, 则$f(x)$是偶函数，$g(x)$是奇函数. 于是

$$\begin{aligned} & \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x\arcsin x+\sin x}{\sqrt{1-x^2}}\mathrm{d}x \\ =& \int_{-\frac{1}{2}}^{\frac{1}{2}}[f(x)+g(x)]\mathrm{d}x \\ =& \int_{-\frac{1}{2}}^{\frac{1}{2}}f(x)\mathrm{d}x+\int_{-\frac{1}{2}}^{\frac{1}{2}}g(x)\mathrm{d}x \\ =& 2\int_0^{\frac{1}{2}}f(x)\mathrm{d}x+0 \\ =& 2\int_0^{\frac{1}{2}}\frac{x\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x \\ =& 2\int_0^{\frac{\pi}{6}}\frac{\sin t\cdot t}{\cos t}\mathrm{d}(\sin t)\quad (x=\sin t) \\ =& 2\int_0^{\frac{\pi}{6}}t\sin t\mathrm{d}t \\ =& -2t\cos t\Big|_0^{\frac{\pi}{6}}+2\int_0^{\frac{\pi}{6}}\cos t\mathrm{d}t \\ =& -\frac{\sqrt{3}}{6}\pi+2\cdot\frac{1}{2} \\ =& 1-\frac{\sqrt{3}}{6}\pi. \end{aligned}$$

□

五、(15分) 设$f(x)$在$[a,b]$连续，$D=\big\{(x,y)\big|x\in[a,b],y\in\mathbb{R}\big\}$, 令

$$g(x,y)=f(x)\sin y,\quad (x,y)\in D.$$

证明：$g(x,y)$在D上一致连续.

证 由连续函数的有界定理知$f(x)$在$[a,b]$有界，故存在常数$M>0$, 使得对任意$x\in[a,b]$, 有$|f(x)|\leqslant M$. 由一致连续性的康托尔定理知$f(x)$在$[a,b]$一致连续，故对任意$\varepsilon>0$, 存在$\delta>0$(不妨设$\delta<\varepsilon$), 使得当$x,x'\in[a,b]$, $|x-x'|<\delta$时，有$|f(x)-f(x')|<\varepsilon$. 于是当$(x,y),(x',y')\in D$, $|(x,y)-(x',y')|<\delta$时，有$|x-x'|<\delta$, $|y-y'|<\delta$, 并且

$$\begin{aligned}
&|g(x,y)-g(x',y')|\\
=\ &|f(x)\sin y-f(x')\sin y'|\\
=\ &|f(x)\sin y-f(x')\sin y+f(x')\sin y-f(x')\sin y'|\\
\leqslant\ &|f(x)\sin y-f(x')\sin y|+|f(x')\sin y-f(x')\sin y'|\\
=\ &|f(x)-f(x')|\cdot|\sin y|+|f(x')|\cdot|\sin y-\sin y'|\\
\leqslant\ &\varepsilon\cdot 1+M\cdot|y-y'|\\
<\ &\varepsilon+M\delta\\
<\ &\varepsilon+M\varepsilon\\
=\ &(M+1)\varepsilon.
\end{aligned}$$

由一致连续的定义知$g(x,y)$在D上一致连续. □

六、(15分) 设$f(x)$在$[0,1]$连续. 证明：

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}\mathrm{d}x = \frac{\pi}{2}f(0).$$

证 令$g(x)=f(x)-f(0)$, 则$g(x)$在$[0,1]$连续，$g(0)=0$. 因为

$$\lim_{n\to\infty}\int_0^1 \frac{n}{1+n^2x^2}\mathrm{d}x = \lim_{n\to\infty}\int_0^n \frac{\mathrm{d}t}{1+t^2}\ (t=nx) = \lim_{n\to\infty}\arctan n = \frac{\pi}{2},$$

所以只需证明

$$\lim_{n\to\infty}\int_0^1 \frac{ng(x)}{1+n^2x^2}\mathrm{d}x = 0.$$

由连续函数的有界定理知$g(x)$在$[a,b]$有界，故存在常数$M>0$, 使得对任意$x\in[a,b]$, 有$|g(x)|\leqslant M$. 由$g(x)$在$[0,1]$连续且$g(0)=0$知对任意$\varepsilon>0$, 存在$\delta>0$(不妨设$\delta<1$), 使得当$x\in[0,\delta]$时，有$|g(x)|<\varepsilon$. 于是有

$$\begin{aligned}
&\left|\int_0^1 \frac{ng(x)}{1+n^2x^2}\mathrm{d}x\right| \\
\leqslant\ & \left|\int_0^\delta \frac{ng(x)}{1+n^2x^2}\mathrm{d}x\right| + \left|\int_\delta^1 \frac{ng(x)}{1+n^2x^2}\mathrm{d}x\right| \\
\leqslant\ & \int_0^\delta \frac{n}{1+n^2x^2}|g(x)|\mathrm{d}x + \int_\delta^1 \frac{n}{1+n^2x^2}|g(x)|\mathrm{d}x \\
<\ & \varepsilon\int_0^\delta \frac{n}{1+n^2x^2}\mathrm{d}x + M\int_\delta^1 \frac{n}{1+n^2x^2}\mathrm{d}x \\
=\ & \varepsilon\int_0^{n\delta} \frac{\mathrm{d}t}{1+t^2} + M\int_{n\delta}^n \frac{\mathrm{d}t}{1+t^2} \\
=\ & \varepsilon\cdot\arctan(n\delta) + M[\arctan n - \arctan(n\delta)] \\
<\ & \frac{\pi}{2}\varepsilon + M[\arctan n - \arctan(n\delta)].
\end{aligned}$$

因为

$$\lim_{n\to\infty}[\arctan n - \arctan(n\delta)] = \frac{\pi}{2}-\frac{\pi}{2} = 0,$$

所以对上述$\varepsilon>0$, 存在正整数N, 当$n>N$时，有$0<\arctan n-\arctan(n\delta)<\varepsilon$. 于是当$n>N$时，有

$$\left|\int_0^1 \frac{ng(x)}{1+n^2x^2}\mathrm{d}x\right| < \frac{\pi}{2}\varepsilon + M\varepsilon = \left(\frac{\pi}{2}+M\right)\varepsilon.$$

按极限定义知

$$\lim_{n\to\infty}\int_0^1 \frac{ng(x)}{1+n^2x^2}\mathrm{d}x = 0.$$

这就完成了证明. □

七、(10分) 设$f(x)$在$[0,1]$连续可微, 且$\int_0^1 f(x)\mathrm{d}x = 0$, 记$M = \max\limits_{0\leqslant x\leqslant 1}|f'(x)|$. 证明：对任意$x\in[0,1]$, 都有

$$\left|\int_0^x f(t)\mathrm{d}t\right| \leqslant \frac{M}{8}.$$

证 令$F(x)=\int_0^x f(t)\mathrm{d}t$, 则$F(x)$在$[0,1]$两次连续可微，$F(0)=F(1)=0$, $M=\max\limits_{0\leqslant x\leqslant 1}|F''(x)|$. 问题归为证明对任意$x\in[0,1]$, 都有$|F(x)|\leqslant \frac{M}{8}$. 由连续函数的最值定理知$|F(x)|$在$[0,1]$取得最大值，设$x_0\in[0,1]$是$|F(x)|$在$[0,1]$的一个最大值点，问题进一步归为证明$|F(x_0)|\leqslant\frac{M}{8}$. 若$F(x_0)=0$, 则$|F(x_0)|\leqslant\frac{M}{8}$显然成立. 若$F(x_0)\neq 0$, 则不妨设$F(x_0)>0$(否则以$-F$代替$F$进行讨论)，于是$x_0\in(0,1)$且$x_0$也是$F(x)$在$[0,1]$的一个最大值点. 由费马定理知$F'(x_0)=0$, 下面分两种情形讨论.

(i) $x_0\in\left(0,\frac{1}{2}\right]$的情形.

由泰勒公式得

$$0=F(0)=F(x_0)+F'(x_0)(0-x_0)+\frac{1}{2}F''(\xi)(0-x_0)^2=F(x_0)+\frac{1}{2}F''(\xi)x_0^2,$$

其中$\xi\in(0,x_0)$. 由此可见

$$|F(x_0)|=\frac{1}{2}|F''(\xi)|x_0^2\leqslant\frac{1}{2}\cdot M\cdot\left(\frac{1}{2}\right)^2=\frac{M}{8}.$$

(ii) $x_0\in\left(\frac{1}{2},1\right)$的情形.

由泰勒公式得

$$0=F(1)=F(x_0)+F'(x_0)(1-x_0)+\frac{1}{2}F''(\eta)(1-x_0)^2=F(x_0)+\frac{1}{2}F''(\eta)(1-x_0)^2,$$

其中$\eta\in(x_0,1)$. 由此可见

$$|F(x_0)|=\frac{1}{2}|F''(\eta)|(1-x_0)^2\leqslant\frac{1}{2}\cdot M\cdot\left(\frac{1}{2}\right)^2=\frac{M}{8}.$$

这就完成了证明. □