一、（本题共 10 分，每小题 5 分）

a) Give the definitions of strictly stationary (强平稳) time series and weakly stationary (弱平稳) time series. Why the stationary condition is important?

b) Give the definition of an integrated autoregressive moving average model (ARIMA). What are the sufficient and necessary conditions for an ARIMA model to be causal and invertible?
Consider the random walk with drift model, \( x_t = \delta + x_{t-1} + w_t, \quad t = 1, 2, \ldots, \) with \( x_0 = 0, \) where \( w_t \) is white noise with mean zero and constant variance \( \sigma_w^2. \)

a) Show that the model can be written as \( x_t = t\delta + \sum_{i=1}^{t} w_i. \)

b) What are the mean function and auto-covariance function of \( x_t? \) Is \( x_t \) stationary?

c) Show the auto-correlation function \( \rho_x(t-1, t) = \frac{\sqrt{t}}{t} \to 1, \) as \( t \to \infty. \)

d) Suggest a transform to make \( x_t \) stationary.
Suppose $x_t$ follows an ARMA(1,1) model with zero mean, $x_t = \phi x_{t-1} + \theta w_{t-1} + w_t$.

a) Obtain the auto-covariance function and auto-correlation function of $x_t$;

b) What is the stationary solution of the (causal) ARMA(1,1) model? Obtain the coefficients for $w_{t-k}$, $k = 0,1,2,\ldots$ in the stationary solution.
四、（本题共 20 分，每小题 10 分）

a) Describe the steps for fitting an appropriate ARIMA model to a time series;

b) For a general stationary time series $x_t$ with mean zero, derive the prediction equations for obtaining the best (minimum-mean-squared-error) linear predictor of $x_{n+m}$ based on $n$ observations, \{x_1, x_2, \ldots, x_n\}.
Consider an AR(1) model, \( x_t - \mu = \phi(x_{t-1} - \mu) + w_t \), where \( \mu \) is a constant mean and \( w_t \) is Gaussian white noise with mean zero and variance \( \sigma_w^2 \). Suppose we observe \( x_1, x_2, \ldots, x_n \) from the \( x_t \) series.

a) Show the Yule-Walker equations for estimating \( \phi \) and \( \sigma_w^2 \). Obtain the corresponding estimate (矩估计) for \( \phi \) based on \( n \) observations;

b) Show the (Gaussian) conditional log-likelihood function (条件对数似然函数) given \( x_1 \) for parameter estimation. Obtain the estimate of \( \phi \) based on the conditional log-likelihood function.
Given data \( x_1, x_2, \ldots, x_n \) with a constant mean \( \mu \), define the discrete Fourier transform for a given frequency \( \omega_j \):

\[
d(\omega_j) = \frac{n^{-1/2}}{\sqrt{n}} \sum_{t=1}^{n} x_t e^{-2\pi i \omega_j t},
\]

where \( i = \sqrt{-1}, \omega_j = \frac{2\pi j}{n} \) and \( j = 0, 1, 2, \ldots, n-1 \).

a) What is the definition of periodogram, \( I(\omega_j) \)? How to interpret it?

b) Show that for \( j \neq 0 \),

\[
I(\omega_j) = \sum_{h=-(n-1)}^{n-1} \hat{\gamma}(h) \exp(-2\pi i \omega_j h),
\]

where \( \hat{\gamma}(h) \) is the sample auto-covariance function. What is the implication of b) for explaining the periodogram \( I(\omega_j) \)?